Villermaux, J., *Génie de la Réaction Chimique. Conception et Fonctionnement des Reacteurs*, Ed., Lavoisier, Technique et Documentation, Paris (1982).
Villermaux, J., "Theory of Linear Chromatography," *Percolation Processes: Theory and Applications*, Eds., A. Rodrigues and D. Tondeur, Sjthoff and Noordhoff, 1 (1981).
Villermaux, J., "Distribution des Temps de Séjour et Mélange Maximal par un Modèle de Mélangeurs en Cascade avec Alimentation Étagée," *Can. J. Chem. Eng.*, **48**, 317 (1970).
Villermaux, J., "Letter to the Editor," *Chem. Eng. Sci.*, **21**, 1054 (1974).
Wen, C. Y., and L. T. Fan, Eds. "Models for Flow Systems and Chemical Reactors," *Chemical Processing and Engineering*, **3**, M. Dekker (1975).
Zoulalian, A., and J. Villermaux, "Influence of Chemical Parameters on Micromixing in a Continuous Stirred Tank Reactor," *Adv. in Chem. Ser.*, **133**, 348 (1974).
Zwittering, T. W., "The Degree of Mixing in Continuous Flow Systems," *Chem. Eng. Sci.*, **11**, 1 (1959).

Manuscript received March 29, 1982; revision received November 30, and accepted December 14, 1982.

Kynch Theory and Compression Zones

BRYANT FITCH
Auburn University
1570 Oak St.
Napa, CA 94559

Kynch's theories of sedimentation are reinterpreted, modified, and extended to be valid for batch sedimentation in which a zone of compacting sediment forms at the bottom of the column. The development has several steps: First, it is shown that a concentration discontinuity, any part of whose chord plots above the curve on a Kynch plot of settling flux vs. particle concentration, will be unstable and immediately give rise to a different concentration distribution. From this it is deduced that Kynch characteristics, or loci of constant concentration, must propagate either from the origin of a height vs. time plot, or tangentially from the locus of the compression or suspension-sediment discontinuity. A Kynch-like construction is derived to relate the settling rate at the top of the suspension (measured by its subsidence rate) to the concentration arriving at the surface at that time. It makes use of two tangents, one to the settling curve (as in Kynch), and another to the locus of the compression discontinuity. Finally a construction, analagous to that of Talmage and Fitch, is deduced for determining required thickener area.

SCOPE

Kynch's well-known theory of sedimentation is based on the assumption or premise that the settling rate at any point in a column of suspension would be a function only of the concentration at that point; i.e., $u = u(c)$. He showed that, where his assumption is uniformly valid, "the relationship between settling rate and particle concentration can be deduced from observations on the fall of the top of the suspension" (in a batch test). But Kynch's assumption is not valid throughout a column in which a zone of compacting sediment forms at the bottom. In the compaction zone itself, u is not dependent on c alone, but also on the solids stress gradient (Michaels and Bolger, 1962; Fitch, 1966, 1975, 1979; Shirato et al., 1970). And just above the compaction zone, the Kynch zones or characteristics do not arise from the origin, as demanded by Kynch theory, but arise from the interface between the compacting sediment and the zone-settling suspension above it (Fitch 1966, 1969; Tiller, 1981). As a consequence, procedures for determining the unit area needed in a Dorr-type thickener, that are based on Kynch theory, are without theoretical basis. In particular, the Talmage and Fitch (1955) procedure for thickener design is unsound.

Tiller (1981) shows that Kynch theory can be corrected or extended to cover the case in which a subjacent compression zone is formed. This is done by taking into consideration the rise of the suspension-sediment interface, as well as the fall of the suspension-supernatant interface. Tiller's procedure, however, is complicated. It involves trial and error selection of a set of characteristic lines such that the integrated amount of solids crossing the suspension-sediment interface up to any time, plus the amount remaining in suspension at that time, equals the amount originally present in the suspension.

The Tiller paper does not recognize or make use of a relationship that follows from Kynch theory: Any characteristic that arises from the compression interface must do so tangentially. Use of this relationship makes it possible to replace the complicated Tiller procedure with a simple geometric construction. And corrections to the Talmage and Fitch construction can be derived.

CONCLUSIONS AND SIGNIFICANCE

Kynch's theory has been generalized and extended to account for the presence of a compaction zone at the bottom of a batch sedimentation column. Means have been given to deduce the relationship between settling rate u and particle concentration c, in the region for which $u = u(c)$, from observations on the fall of the top of the suspension and the rise of the compression interface. The rise of the compression interface, however, is not visible. It could in principle be observed with the aid of radiation absorption instruments. It can also be plotted from the observed compression points of a series of batch settling tests having the same concentration but different initial heights. The latter, however, requires multiple tests, which negates the principal practical advantage of Kynch-type procedures for specifying thickener unit area. The method would, therefore, seem to be of greater theoretical significance than of practical utility.

Deduction of a valid theory increases understanding of the sedimentation process for flocculent suspensions. And it provides a basis for evaluating currently-used but theoretically unfounded Kynch-type procedures for thickener design. Of particular interest is the fact that it gives a rationale and a partial theoretical justification for the widely-used but previously wholly empirical Oltmann procedure for determining thickener unit area. It also makes clear certain limitations of that procedure.

KYNCH THEORY AND COMPRESSION ZONES

Kynch theory rests on three theorems and their interpretations. We reinterpret his theorems and develop the theory in a generalized way that accounts for compaction zones and remains valid where they are formed.

Kynch Theorem I

If there is a discontinuity propagating in a sedimentation column, it will do so at a velocity v equal to $-\Delta S/\Delta c$.

$$v = -\Delta S/\Delta c \quad (1)$$

Theorem I is not dependent on the assumption that $u = u(c)$. It is valid for any suspension in which S and c can be defined above and below a discontinuity, provided the discontinuity propagates stably. In particular, it will be valid for a discontinuity between zone-settling suspension above and compacting sediment below, even though in the sediment below $u \neq u(c)$. The stability of discontinuities will be discussed later.

If u and hence also S are functions of c, a plot of S vs. c is meaningful, Figure 1. If there is a discontinuity with concentration c_1 above and c_2 below, $\Delta S/\Delta c$ is the slope of the chord between points (1) and (2) on the plot. The propagation velocity v of the discontinuity will be the negative of this slope.

As will be shown, a discontinuity whose chord lies completely below the flux plot will (under the assumptions of the model) be stable. A discontinuity, as between c_1 and c_3, any part of whose chord lies above the flux plot, will be unstable, and will immediately give rise to some other concentration distribution.

Kynch Theorem II

Where $u = u(c)$, the locus of any constant concentration c_a in a concentration gradient will propagate upward at a constant velocity β equal to $-dS/dc$ evaluated at c_a.

$$\beta = -dS/dc \quad (2)$$

These Kynch zones of constant concentration are, under the assumption that $u = u(c)$, characteristics of a continuity equation,

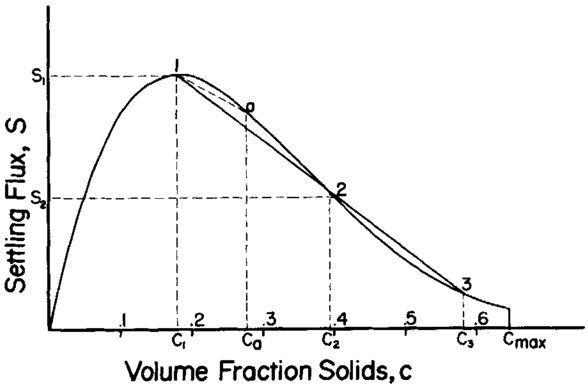

Figure 1. Kynch discontinuities on flux plot.

Kynch Theorem III

The concentration c_a for any characteristic propagating from the origin in a batch settling test, and arriving at the pulp-supernatant interface at time t_a, will be:

$$c_a = \frac{c_o H_o}{(u_a + \beta_a)t_a} \quad (3)$$

Theorem III is usually represented geometrically, since $(u_a + \beta_a)t_a$ is the H intercept of a tangent drawn to the settling plot at point (a), at which the concentration reaches the interface. It is not valid for concentrations whose characteristics arise from the compression discontinuity at the top of the zone of compacting sediment. A construction that remains valid in such cases will be the central result of this paper.

Another assumption Kynch makes in developing his theory is that the settling velocity u tends to zero as $c \rightarrow c_{max}$. He makes this "to insure a simple mathematical presentation." However, he recognizes the invalidity of this assumption. He goes on to say it "cannot always be true in so far as a thick layer of c_{max} above a layer of pure liquid must gradually fall through it or disperse. This is not an important difficulty, observes Kynch, as we can assume that u tends to a small but finite value as $c \rightarrow c_{max}$ and then decreases suddenly to zero."

The point is important, because Kynch's "resulting discontinuity in S," together with the corresponding discontinuity in u, cannot be ignored. It is responsible for the "compression point" in batch subsidence tests, and lies at the heart of the interaction between zone-settling and subjacent compaction zones.

In zone settling, the unbuoyed weight of settling solids is borne hydrodynamically by resistance to counterflowing fluid. That is:

$$g(\rho_s - \rho_f)c = uk \quad (4)$$

where k is the specific hydraulic resistance to flow of fluid, and is considered to be a continuous function of c. Or:

$$u = \frac{g(\rho_s - \rho_f)c}{k(c)} \quad (5)$$

Specific resistance $k(c)$ does not become infinite at either the packing density c_{max} of incompressible sediments, or at the "final dilution" of compressible ones. Therefore, u does not approach zero as c (for incompressible sediments) approaches c_{max}. It approaches rather some finite value corresponding to c_{max}, which Shannon et al. (1963) find to be 0.64 for glass spheres. And this leads to a flux plot of the shape shown in Figure 1. It thus gives rise necessarily to a discontinuous version of the "two-humped" or "doubly-concave flux plot" of Tory (1961) and Shannon et al. (1963). Kynch shows a doubly-concave flux plot as "within the bounds of possibility," but does not analyze it other than to note: "If the preceding methods are applied, three modes of settling are to be found."

STABILITY OF DISCONTINUITIES

It is questionable that any discontinuity with a free-settling concentration below will propagate completely unchanged in a real column, because inertial effects cannot be negligible across a discontinuity, no matter how slowly the particles may be settling

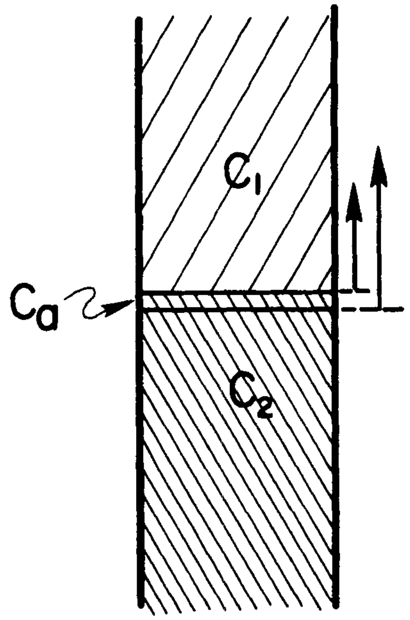

Figure 2. Perturbation on Kynch discontinuity.

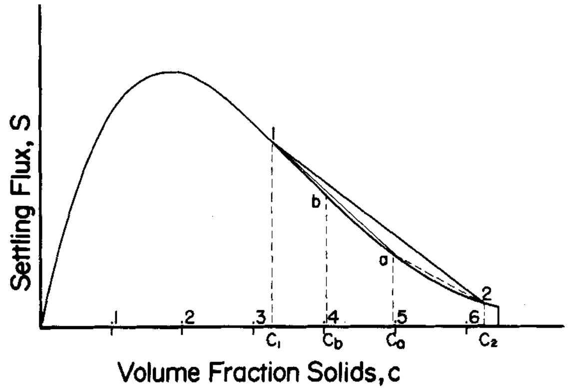

Figure 3. Construction for discontinuity stability.

(Dixon, 1977). But it is possible under the Kynch model. What we now consider are gross instabilities to perturbations in concentration at a discontinuity, under the assumptions of the Kynch model.

First, a discontinuity whose chord lies everywhere below the flux plot will be stable against such perturbations. Consider a column with concentrations c_1 above a discontinuity, and c_2 below it. And assume that the corresponding chord lies everywhere below the flux plot, as chord (1)-(2) in Figure 1. If by perturbation or for any other reason, a concentration c_a between c_1 and c_2 appeared at the discontinuity, the initial discontinuity would be divided into two, one between c_1 and c_a above, and one between c_a and c_2 below (Figure 2). The upward propagation rate of the upper discontinuity would be measured by the negative slope of the chord between points (1) and (a) (Figure 1). That of the lower discontinuity would be measured by the negative slope between points (a) and (2). The latter is greater. The lower discontinuity would propagate upward faster than the upper one. The lower discontinuity would thus overtake the upper one, and would merge with it to recreate the original one between points (1) and (2), thus wiping out the perturbation zone of concentration c_a. A discontinuity whose chord lies everywhere below the flux plot will, therefore, under the Kynch assumptions, propagate stably.

Second, a discontinuity whose chord lies everywhere above the flux plot (Figure 3) will be unstable to such perturbations. If an intermediate concentration c_a appeared at the discontinuity, the resulting upper discontinuity, between c_1 and c_a, would propagate upward faster than the one between c_a and c_2 below it. The intermediate zone of concentration c_a would then grow. But the chords for both resulting discontinuities also run above the flux plot. Further perturbations, as to produce some concentration c_b between c_1 and c_a, would lead to the growth of a zone of c_b. But the resulting chords still run above the flux plot. So by accumulated perturbations the range would be broken down into smaller and smaller discontinuities, each propagating upward faster than those between higher concentrations below them in the column. In the limit, the individual discontinuities would become infinitesimal. The final result is a region of graded concentration or a concentration gradient between c_1 and c_2. Each infinitesimal concentration step corresponds to a Kynch characteristic, rising out of the original discontinuity and propagating upward through the gradient region at its characteristic velocity. Thus, as shown by Kynch, an initial discontinuity whose chord lies everywhere above the flux plot will give rise to a concentration gradient.

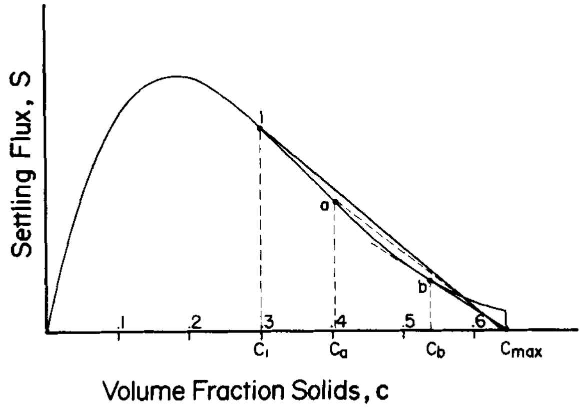

Figure 4. Construction for discontinuity stability.

An initial discontinuity between some concentration c_1 (Figure 4) and a rising, incompressible sediment at c_{max}, will now be considered. The solids in the sediment at c_{max} have a settling velocity of zero, so the corresponding point on the flux plot will lie on the c axis at c_{max}. A chord from (c_{max}) will be tangent to the flux plot at some point (b). If any concentration c_a between c_1 and c_b arises at the interface its zone will grow, because the resulting discontinuity between c_a and c_{max} propagates upward more slowly than the one between c_1 and c_a above it. But the slowest lower discontinuity that can form is that between c_b and c_{max}. It is stable, since its chord lies everywhere below the flux plot, and it cannot overtake and merge with any other possible discontinuity, since it is the slowest. The upper discontinuity between c_1 and c_b, on the other hand, is unstable, since its chord lies everywhere above the flux plot. It will, therefore, give rise to a zone of graded concentration. Thus, the original discontinuity gives rise to a zone of graded concentration from c_1 to c_b, above a compression discontinuity from c_b to c_{max}. And the Kynch characteristic for c_b, which is the most concentrated characteristic in the graded zone above the compression discontinuity, propagates upward at the same rate as the discontinuity itself, since the discontinuity chord is tangent to the flux plot at c_b.

In a continuum, one could not predict when the perturbations will arise. But in a particulate system a true discontinuity could not exist. To settle from one concentration to a higher one, particles must come progressively closer over a finite distance. Over this distance a finite $\partial c/\partial x$ exists, and thus every concentration between that above and that below the putative discontinuity. Zones of intermediate concentration form immediately, and one does not have to wait for perturbations. All the Kynch characteristics whose origin is attributed in continuum theory to perturbation will originate in a very short period of time. They will thus appear as a bundle originating at zero time.

Kynch recognized the above. He considers two cases, each having a discontinuity from c_1 above to c_2 below. In one case (his Figure 3c) the discontinuity is treated as the limit of a very rapid change from c_1 to c_2. This immediately gives rise to a bundle of characteristics, the slowest one having an upward propagation velocity equal to that of the resulting discontinuity between it and a zone of c_2 below. In the other case the change from c_1 to c_2 is treated as a mathematical discontinuity. In such a case delayed rising of characteristics out of whatever discontinuity persists up to that time, at a propagation rate greater than that of the discontinuity (his Figure 3d) is mathematically possible. Equations for such characteristics can satisfy continuity and the initial conditions. But as Kynch notes, they are "physically impossible solutions."

BATCH SETTLING, COMPRESSIBLE SOLIDS

Solids at the top of a compression zone will always have the concentration c_d at which the solids structure first exhibits compressive yield value. They could not have a lower concentration, because no lower concentration would support compressive solids stress. They could not have a higher concentration, because there are no solids above them not totally supported by hydraulic forces. Hence there is no compressive stress at the top of the compression zone to overcome the yield value and squeeze the solids to concentrations higher than c_d.

If the solids are compressible, the subsidence velocity just below the compression discontinuity will not be zero, except at time zero, when the depth of the compaction zone is also zero. At later times they will have a nonzero subsidence velocity, as a result of compaction of the structure and squeezing out of fluid from below them. Since the rate at which liquid is expressed from the compaction zone increases as long as the zone is building up, u_d must correspondingly be increasing. Therefore, S_d is also increasing.

Kynch's theorem I may be expressed for this case as:

$$v = \frac{S_c - S_d}{c_d - c_c}$$

Now c_d is, as has been shown, constant. Both S_c and c_c vary slowly, as different concentration characteristics arise above the discontinuity. But the effect of S_d is dominating. Therefore, the compression discontinuity arises with a velocity v that decreases with time, and hence with a decreasing slope dL/dt in a settling plot: (curve (0)-(c) in Figure 5). A flux plot for this case is shown in Figure 6. The concentration at the top of the compression zone, hence just below the compression discontinuity, will be c_d. At some particular time t_e, (Figure 5) the flux will be S_e. The corresponding point on the flux plot (Figure 6) is (e). A chord from point (e) in Figure 6 will be tangent to the flux plot at some point (f). Thus chord (e)-(f) represents a compression discontinuity stable at that time, and c_f is the lowest concentration that can form above the discontinuity. Any lower one would yield a graded zone above a c_f-c_d discontinuity.

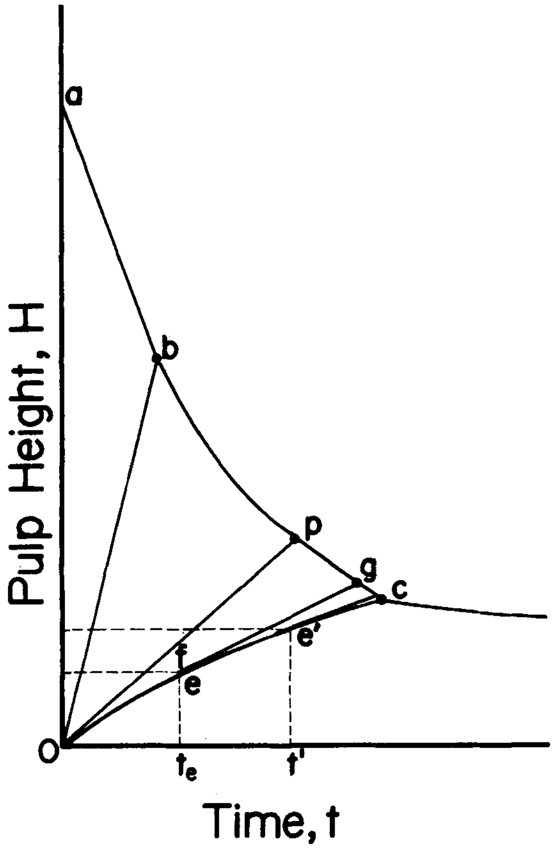

Figure 5. Batch settling curve, compressible solids.

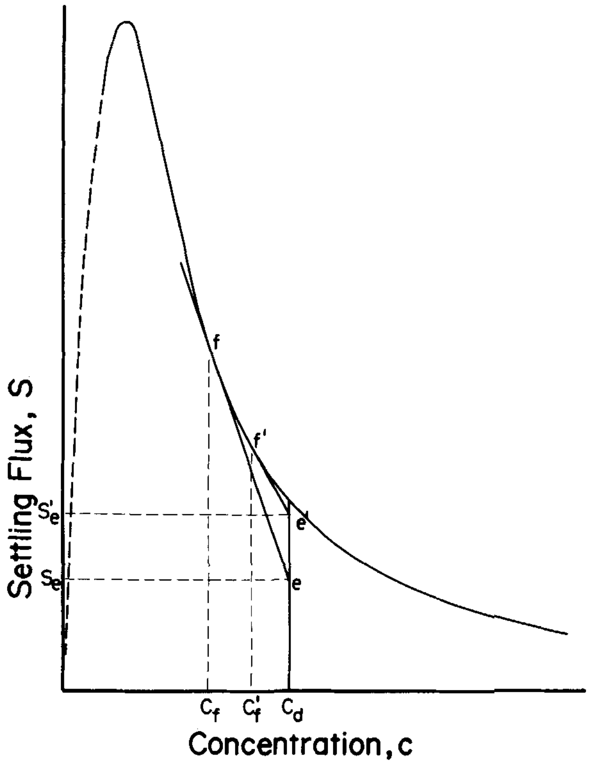

Figure 6. Compression discontinuities, compressible solids.

At some later time t' the subsidence velocity of particles at the top of the compression zone would be higher, corresponding to some subsidence flux S_e' higher than S_e. The corresponding chord from (e') (Figure 6) would be tangent to the flux plot at some concentration (f'). If the discontinuity (c_f)-(c_d) had persisted up to this time, it would break down into a graded zone from c_f to c_f' above a compression discontinuity from c_f' to c_d. And the characteristic for c_f' (rising from e' in Figure 5) would be tangent to the discontinuity locus (have the same rising rate as the compression discontinuity) at that time. The closer c_f' is to c_f, the smaller the range of concentrations in the graded zone. And as c_f' approaches c_f, or in the limit as the compression discontinuity locus becomes continuous, the only characteristic left to arise from the compression discontinuity at $t + dt$ is that for c_f' itself. Therefore any characteristic arising from the compression discontinuity must do so tangentially (Fitch, 1966, 1969).

KYNCH THEOREM III (GENERALIZED)

The amount of solids per unit area, Q, above a Kynch characteristic at any time t_1 will be:

$$Q_1 = c_a(u_a + \beta_a)(t_a - t_1) \qquad (6)$$

Where t_a is the time at which the characteristic reaches the pulp-supernatant interface.

This is proven as follows: The volumetric flux of solids settling at velocity u_a past a characteristic locus rising at velocity β_a, is $c_a(u_a$

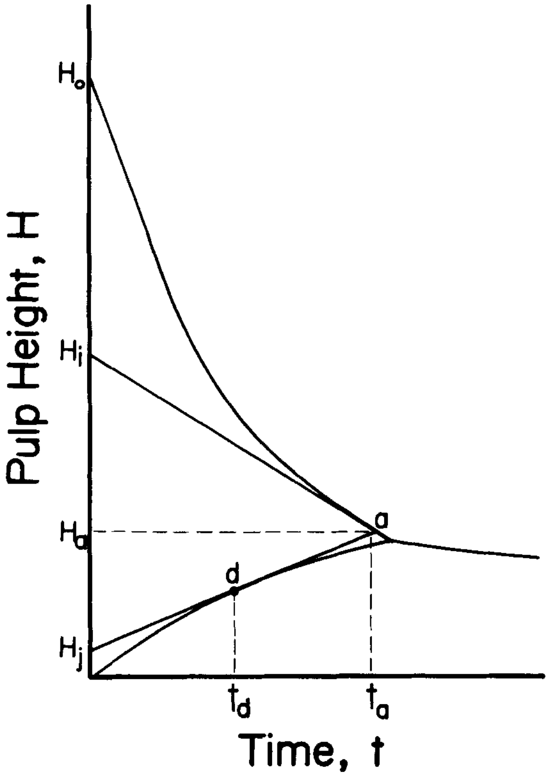

Figure 7. Generalized Kynch-type construction.

$+ \beta_a)$. Whatever solids are above the locus at time t_1 will have passed beneath it at time t_a. The Q_1 passing the locus in the time interval $(t_a - t_1)$ is the solids flux times the interval, or:

$$Q = c_a(u_a + \beta_a)(t_a - t_1) \tag{7}$$

Kynch applied this proof to the special case that t_1 coincided with the time sedimentation started in a batch test originally at uniform concentration, hence when $t = 0$ in a settling plot. Since $(u_a + \beta_a)t_a = H_i$, and for Kynch's special case $Q_o = c_oH_o$:

$$c_a = \frac{c_oH_o}{H_i} \tag{8}$$

Tiller (1981) extends application of Kynch's proof to the case that t_1 coincides with the time a characteristic arises from the compression discontinuity. He thus points the way to more complete generalization.

Kynch's construction, together with Eq. 8, can be used to determine the concentration at the surface for the Kynch regime of a settling curve for concentrations whose characteristics originate at the bottom of the column at time zero. They cannot be used for the transition section, whose characteristics arise from the compression discontinuity at later times.

A corrected construction, Figure 7, allowing for the transition zone, is derived in the following. For any point (a) in the transition region of a batch settling curve, draw a tangent to the settling curve at point (a) (as in the Kynch procedure). It will intercept the H axis at some point (H_i). Draw also a tangent to the compression interface curve (0)-(d)-(c) from point (a). It will intercept the H axis at some point (H_j). The Kynch characteristic for concentration c_a is the segment (d)-(a), but the position of point (a) and the slope dH/dt of the settling curve at (a) is the same as though a characteristic had propagated from point (H_j), and all the solids originally above the characteristic had crossed it by the time it had propagated to point (a). The amount of solids originally above this partially hypothetical characteristic would be $c_o(H_o - H_j)$, or:

$$Q_j = c_o(H_o - H_j) \tag{9}$$

The solids flux past the characteristic will be $c_a(\beta_a + u_a)$. Therefore the quantity of solids passing it between t_o and t_a will be the flux times t_a

Or:

$$Q_j = c_a(\beta_a + u_a)t_a \tag{10}$$

But geometrically:

$$\beta_a = \frac{H_a - H_d}{t_a - t_d} = \frac{H_a - H_j}{t_a}$$

$$u_a = \frac{H_i - H_a}{t_a}$$

Therefore, substituting in Eq. 10

$$Q_j = c_a(H_a - H_j + H_i - H_a)$$
$$Q_j = c_a(H_i - H_j) \tag{11}$$

Equating Eqs. 9 and 11

$$c_a(H_i - H_j) = c_o(H_o - H_j)$$
$$c_a = c_o\left[\frac{H_o - H_j}{H_i - H_j}\right] \tag{12}$$

Therefore, the construction of Figure 7 (to determine H_i and H_j), together with Eq. 12, give the concentration just under the suspension-supernatant interface at point (a).

When a characteristic drawn from point (a) does not fall tangent to the compression curve, it is drawn to the origin, and the construction reduces to that of Kynch.

Note that the construction of Figure 7 may be considered as a Kynch theorem III construction in a coordinate system whose origin is at point (H_j).

The construction of Figure 7 requires plotting of the compression discontinuity as well as that of the top of the suspension during batch settling. The compression discontinuity is not visible in a batch settling test, but could be monitored with a density scanning device (Gaudin and Fuerstenau, 1958). It could also be found by plotting compression points for a series of batch settling tests made at the given initial concentration, but at various initial heights (Figure 8).

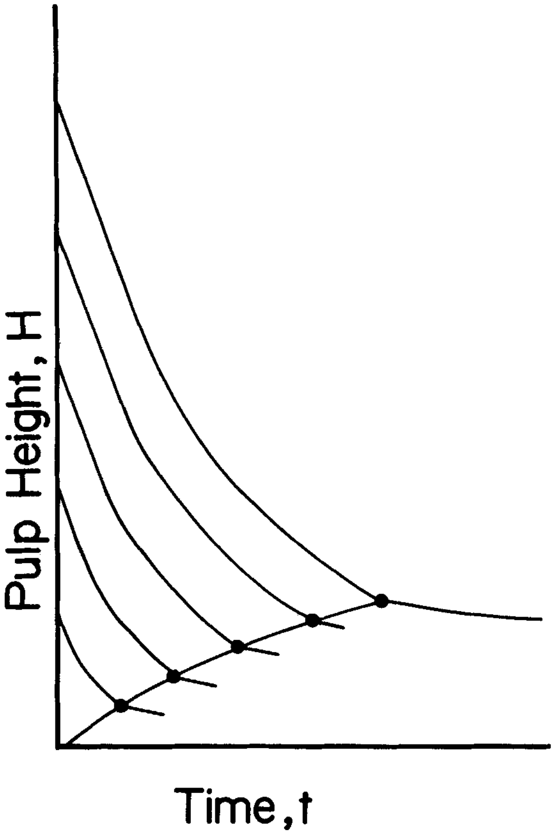

Figure 8. L, or locus of compression discontinuity, from tests at different values of H_o.

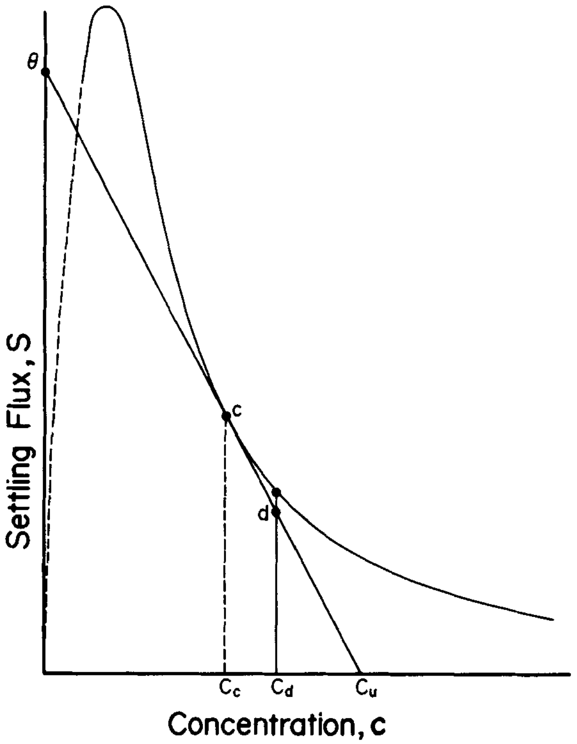

Figure 9. Yoshioka construction for continuous thickening, showing corresponding batch compression discontinuity.

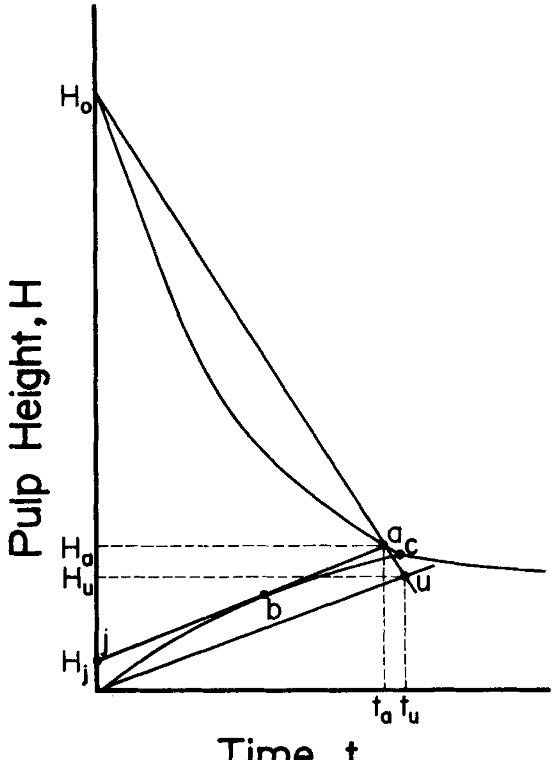

Figure 10. Corrected construction for thickener unit area.

CONTINUOUS THICKENING

Continuous or steady-state thickening can be represented in a flux plot by a Yoshioka construction (Yoshioka et al., 1957). An "operating" or material balance line is drawn from c_u on the c axis, tangent to the under side of the flux plot (Figure 9). This operating line will intersect the S axis at a value equal to θ, the maximum possible thickener flux that will yield c_u in the underflow. The point of tangency (c) identifies the critical zone. And operating line segment (c)-(d) represents the compression discontinuity, with c_c above and c_d below.

The segment of the operating line from (c) to (d) can also represent the compression discontinuity in a batch test, just as a characteristic corresponding to c_c arises from the compression discontinuity (if there are enough solids initially present to permit such a characteristic to arise). Looked at another way, the compression discontinuity at any particular time in a batch test corresponds to some steady-state operation. We will now show how to determine the c_u and θ for the corresponding steady-state operation, that is, a continuous operation having the same compression discontinuity, from the compression discontinuity existing at a particular time in a batch test.

The slopes of the operating line, the discontinuity chord (c)-(d), and of the flux plot dS/dc at the critical concentration are identical. The slope of the operating line is $-\theta/c_u$, and since $\theta = c_u v_u$, the slope is $-v_u$. The propagation rate of the compression discontinuity is the negative slope of the chord (c)-(d), and the propagation rate of the critical zone above the discontinuity is $-dS/dc$ evaluated at c_c. Therefore:

$$\beta_c = v_c = v_u$$

This is not surprising. In steady state, continuous thickening the locus of any compression discontinuity (if it exists) must remain stationary. And such a discontinuity must exist if the feed has a concentration below c_d, the lowest concentration at which a compressive yield value exists. The upward propagation v_c of such a discontinuity with respect to the particulate system must be equal to the downward velocity v_u imparted to the system as a whole by underflow withdrawal. And the propagation rate v_c of the discontinuity equals β_c, that of the critical concentration just above it.

The construction to determine thickener area is shown in Figure 10, which is a plot of the suspension-supernatant interface (H_o)-(a)-(c), and of the locus of the compression discontinuity (0)-(b)-(c), as a function of settling time, in a batch test.

Some point (a) in the transition section of the settling curve is chosen. Its concentration c_a is as yet unknown, but could be calculated by the procedures given earlier. Construct the characteristic for c_a, through point (a) and tangent to the compression curve at some point (b). Extension of the characteristic line will intercept the H axis at (H_j). Draw a second construction line parallel to the characteristic and passing through the origin. Draw another construction line from (H_o) on the H axis, through point (a). It intersects the second construction line at time t_u and height H_u.

At point (b) the compression discontinuity has a propagation rate:

$$\left.\frac{dL}{dt}\right|_b = v_b = \frac{H_a - H_j}{t_a} \tag{13}$$

The flux of solids past the discontinuity at point (b) is the same as if all solids originally present above (H_j) had crossed the discontinuity in time t_a, at a rate they are crossing it at point (b). The amount of solids originally above (j) would be:

$$Q_j = (c_o H_o)\left[\frac{H_o - H_j}{H_o}\right] \tag{14}$$

And if they crossed the discontinuity at a constant rate in time t_a their flux would be:

$$\theta = Q_j/t_a = \frac{c_o H_o}{t_a}\left[\frac{H_o - H_j}{H_o}\right] \tag{15}$$

But from geometry:

$$t_u = t_a\left[\frac{H_o}{H_o - H_j}\right] \tag{16}$$

Substituting Eq. 16 in Eq. 15

$$\theta = \frac{c_o H_o}{t_u} \tag{17}$$

The underflow flux is also:

$$\theta = c_u v_u = c_u v_b = c_u \left[\frac{H_a - H_j}{t_a}\right] \tag{18}$$

But from geometry:

$$\frac{H_a - H_j}{t_a} = \frac{H_u}{t_u} \tag{19}$$

So:

$$\theta = c_u \left[\frac{H_u}{t_u}\right] \tag{20}$$

Equating Eqs. 17 and 20:

$$\frac{c_u H_u}{t_u} = \frac{c_o H_o}{t_u}$$
$$c_u = \frac{c_o H_o}{H_u} \tag{21}$$

Equations 17 and 21 give respectively the flux θ and the underflow concentration c_u corresponding to the choice of point (a), and hence concentration c_a, as the critical one.

LIMITATIONS

The procedure proposed chooses a critical concentration. This concentration is not evaluated explicitly, although it could be by Eq. 12. But the c_u and θ corresponding to this critical concentration are evaluated indirectly through implicit interrelations. In particular, c_u may not then be specified independently.

The method is not valid unless there is a Kynch and/or a transitional falling rate segment of the batch settling plot. There must be Kynch characteristics propagating to the surface before the compression point is reached. In the limit, as the initial height of column appoaches zero, Kynch characteristics will not appear if the feed concentration is below c_f, Figure 11. Since the chord (f)-(c_d) is everywhere below the flux plot, no zone of graded concentration will arise from it initially. But as greater initial heights permit buildup of a compression zone, the subsidence rate of particles below the compression interface will no longer be zero.

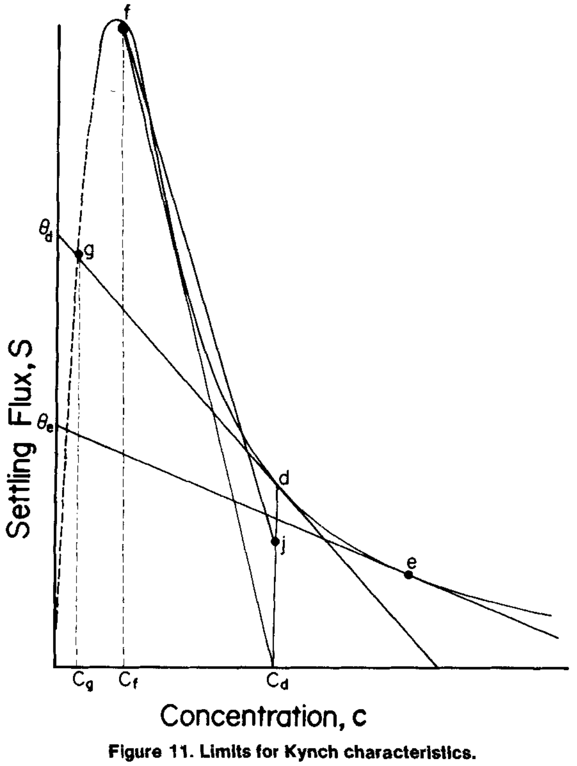

Figure 11. Limits for Kynch characteristics.

The chord from (f) to some corresponding point (j) will pass above the flux plot, and Kynch characteristics will then arise.

At some feed concentration c_g, the chord (g)-(d) becomes tangent to the flux plot at c_d. Point (d) marks an upper bound for the subsidence rate of solids at the top of a compression zone, since they are then subsiding at their free settling rate, and solids stress ψ is zero. Therefore at any initial concentration below c_g, no characteristic will arise, no matter how great H_o and the corresponding depth of compression zone that can arise.

There are also continuous operations that are apparently possible which cannot be predicted by the method. They arise when the critical concentration lies within the compression range of concentrations, such as c_e in Figure 11. In such a case there is no critical concentration just above the compression interface in continuous operation, to correspond with the Kynch characteristics rising from the interface in batch operations. And in such cases the derivations given above would be invalid.

Thus for the method to be valid, Kynch characteristics must arise from the compression discontinuity in the batch test, and the concentration just above the discontinuity in continuous operation must be the critical one.

Note that the method tells what underflow solids concentration corresponds to a given critical one, but tells nothing about how deep the compression zone must be in continuous operation to attain the corresponding c_u. That is, it does not give any information on how deep the continuous compression zone would have to be to make the concentration critical.

ACKNOWLEDGMENT

The author gratefully acknowledges support from the National Science Foundation.

NOTATION

c	= volume fraction solids
c_c	= volume fraction solids above a compression discontinuity (constant when sediment is incompressible)
c_d	= volume fraction solids at top of compression zone
c_o	= initial volume fraction solids
c_{max}	= packing volume fraction for incompressible solids
c_u	= underflow concentration
g	= acceleration of gravity
H	= height above bottom of column
H_o	= initial height of suspension in batch sedimentation
H_i	= height determined by construction on plot of H vs. t for batch settling
H_j	= height determined by construction on plot of L vs. t for batch settling
k	= specific hydraulic resistance of solids structure considered as a porous medium
L	= height of compression discontinuity above bottom of column
Q	= volume of solids per unit area in suspension above a compaction zone
Q_o	= value of Q at time zero
S	= solids volumetric settling flux = uc
t	= time
u	= settling rate of particle
u_c	= settling rate of particles just above a compression discontinuity
u_d	= subsidence rate of solids at top of compression zone at a suspension-sediment discontinuity
v	= upward propagation rate of discontinuity
v_c	= upward propagation rate of compression discontinuity
v_u	= downward velocity of pulp in continuous thickening resulting from underflow withdrawal
x	= distance down column; $dx = -dH$

Greek Letters

β = upward propagation rate of Kynch characteristics, or locus of constant concentration in a concentration gradient
θ = thickener flux, volume of solids per unit time per unit area moving down thickener
ρ_f = density of fluid
ρ_s = density of solids
ψ = solids pressure

LITERATURE CITED

Dixon, D. C., "Momentum-Balance Aspects of Free-Settling Theory," *Separation Sci.*, 12(2), 171, 193 (1977).
Fitch, B., "Biological Treatment of Sewage and Industrial Wastes," **2**, 159, Reinhold, New York (1958).
Fitch, B., "Sedimentation Process Fundamentals," *Trans. Amer. Inst. Mining Eng.*, **223**, 129 (1962).
Fitch, B., "Current Theory and Thickener Design," *Ind. & Eng. Chem.*, **58**, 18 (1966).
Fitch, B., "Current Theory and Thickener Design, 1975," *Filtration and Separation*, **12**, 355, 480, 636 (1975).
Fitch, B., "Sedimentation of Flocculent Suspensions, State of the Art," *AIChE J.*, **25**, 913 (1979).
Gaudin, A. M., and M. C. Fuerstenau, "The Transviewer-X Rays to Measure Suspended Solids Concentration," *Eng. Min. J.*, **159**, 110 (Sept. 1958).
Kynch, G. J., "A Theory of Sedimentation," *Trans. Faraday Soc.*, **48**, 166 (1952).
Michaels, A. S., and J. C. Bolger, "Settling Rates and Sediment Volumes of Flocculated Kaolin Suspensions," *I. & E.C. Fund.*, **1**, 24 (1962).
Shannon, P. T., E. Stroupe, and E. M. Tory, "Batch and Continuous Thickening," *I. and E.C. Fund.*, **2**, 203 (1936).
Shirato, M., Kato, K. Kobayashi, and H. Sakazaki, "Analysis of Settling of Thick Slurries Due to Consolidation," *J. Chem. Eng. Japan*, **3**, 98 (1970).
Talmage, W. P., and E. B. Fitch, "Determining Thickener Unit Areas," *I. & E.C.*, **47**, 38 (1955).
Tiller, F. M., "Revision of Kynch Sedimentation Theory," *AIChE J.*, **27**, 823 (1981).
Tory, E. M., "Batch and Continuous Thickening of Slurries," PhD Thesis, Purdue University (1961).
Yoshioka, N., Y. Hotta, S. Tanaka, S. Naito, and S. Tsugami, "Continuous Thickening of Homogeneous Flocculated Slurries," *Kagaku Kogaku*, **21**, 66 (1957).

Manuscript received June 5, 1981; revision received July 6, and accepted August 26, 1982.

Forced Convection in Three-Dimensional Flows:

III. Asymptotic Solutions with Viscous Heating

Matched asymptotic solutions are given for the temperature profiles and heat transfer rates in laminar three-dimensional flows with viscous dissipation. The results are asymptotically valid for small thermal diffusivity α; they hold for Newtonian or non-Newtonian fluids. The fluid properties are evaluated at an average temperature for the system; this is satisfactory for moderate temperature differences. Heat transfer formulas for various thermal boundary conditions are included. Several examples are analyzed, including the frictional heating of a polymer melt in flow through a wire-coating die.

W. E. STEWART and M. A. McCLELLAND

Department of Chemical Engineering
University of Wisconsin
Madison, WI 53706

SCOPE

Small-diffusivity asymptotes give useful understanding of heat and mass transfer operations. Parts I and II of this series (Stewart, 1963; Stewart, Angelo and Lightfoot, 1970) dealt with boundary-layer phenomena in the absence of viscous heating. The present paper adds the effects of viscous heating in the boundary layers and in the bulk of the fluid. The results are relevant to wire-coating, extrusion, and other flow processes involving heat transport with rapid deformation or highly viscous fluids.

This treatment holds for laminar nonseparated regions of external flows, and for laminar nonseparated thermal entrance regions of internal flows. The velocity profiles are considered as given or separately calculable. The physical properties are treated as constants, evaluated at an average temperature for the given problem.

The superposition principle is used extensively in this paper. The temperature profiles are derived as sums of dissipation and heat-transfer contributions. The dissipation rate itself can be expressed as a sum of simpler functions. By a special choice of such functions, we obtain new similarity solutions which are superimposed to solve Eq. 18.

Parts I and II of the series have been published in *AIChE J.* in 1963 and 1970, respectively. See the Literature Cited section for complete information.